MATH5010 Linear Analysis (2022-23): Homework 5. Deadline: 06 Nov 2022

Important Notice:

♣ The answer paper must be submitted before the deadline.
♠ The answer paper MUST BE sent to the CU Blackboard.

1. Let $X := \{f : [a,b] \to \mathbb{R} : f \quad \text{is continuous on} \quad \mathbb{R}\}$. For each $f \in X$, let $\|f\|_1 := \int_a^b |f(t)|dt$ and $\|f\|_\infty := \sup\{|f(t)| : t \in [a,b]\}$. Put

$$Tf(x) := \int_a^x f(t)dt$$

for $x \in [a,b]$.

   (i) Show that $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_\infty)$ is a bounded linear map of norm 1.

   (ii) Show that $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_1)$ is a bounded linear map of norm $b-a$.

2. Let X be a normed space over $\mathbb{C}$. Let $x, y \in X$ such that $\|x-y\| > c > 0$. Show that there is an element $f \in X^*$ such that $f(x) > c + f(y)$.

3. Assume that the vector space $\mathbb{C}^m$ is endowed with the usual norm, i.e., $\|(z_1, ..., z_m)\| := \sqrt{|z_1|^2 + \cdot + |z_m|^2}$. Define a map T from $\mathbb{C}^m$ to its dual space by $Tz(w) := \sum_{k=1}^m z_k w_k$ for $z = (z_1, ..., z_m)$ and $w = (w_1, ..., w_m)$ in $\mathbb{C}^m$. Show that the map T is an isometric isomorphism.

$* * *$ **End** $* * *$